Quiz 2

1. If $t(n)$ is a function such that $t(1) = 1$ and $t(n+1) \leq 3t(n) + 1$ then which of the following is true

   (a) There exists a $c > 0$ and N_0 such that $t(n) \leq c^n$ for all $n \geq N_0$

   (b) There exists a $c > 0$ and N_0 such that $t(n) \geq 3^{cn} n$ for all $n \geq N_0$

   (c) For all $c > 0$ there exists N_0 such that $t(n) \geq n^c$ for all $n \geq N_0$

   (d) For all $c > 1$ and there exist $N_0 \geq 0$ such that $t(n) \leq c3^n$ for all $n \geq N_0$.

   Ans: a and d. One may guess $t(n) \leq \sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2}$ and prove it by induction. Hence,

   (a) True for c = 3

   (b) False as, if t(n) = 1, that is, it is a constant sequence, the statement does not hold

   (c) False, Same as above

   (d) True, as $t(n) \leq 3^n \leq c3^n \; \forall c > 1$

2. Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined by $b_1 = 3, b_2 = 6, b_k = b_{k-2} + b_{k-1}$ for $k \geq 3$. Prove that b_n is divisible by 3 for all integers $n \geq 1$. Regarding the induction hypothesis, which is true?

   (a) Assuming this statement is true for $k \leq n$ is enough to show that it is true for $n+1$ and no weaker assumption will do.

   (b) Assuming this statement is true for n and $n-1$ is enough to show that it is true for $n+1$.

   (c) Assuming this statement is true for $n, n-1$, and $n-3$ is enough to show that it is true for $n+1$ and no weaker assumption will do since you need three consecutive integers to insure divisibility by 3.

   (d) Assuming this statement is true for n is enough to show that it is true for $n+1$.

   (e) Assuming this statement is true for n and $n-3$ is enough to show that it is true for $n+1$ since 3 divides n if and only if 3 divides $n-3$.

   Ans: b.

   (a) False, as assumption in (b) is weaker but is yet sufficient

   (b) True, as $3|b_n$ and $3|b_{n-1} \implies 3|b_{n+1}$. This is necessary and sufficient

(c) False, as assumption in (b) uses only two integers, not necessarily consecutive to prove the claim

(d) False, as it is not sufficient as $b_{k+1} \equiv b_k + b_{k-1} \pmod 3$

(e) False, as the statement is clearly wrong

3. If $P(n)$ is "$2^n > (n)^2$" note that $P(k) \implies P(k+1)$ for all $k \geq 3$. That means:

   (a) $P(n)$ is true for all $n \geq 3$

   (b) $P(n)$ is true for all $n \geq 4$

   (c) $P(n)$ is true for all $n \geq 5$

   (d) $P(n)$ is not true for any n.

   Ans: c $n \geq 5$

   (a) False, counterexample: 3

   (b) False, counterexample: 4

   (c) True, can be proved by induction

   (d) False, as (c) is True

4. If P(n) is "$2n < n!$" then P(n) is true for all $n \geq x$. Then x=? (Give the best possible integer answer, i.e. if it is true for x=7 and x=15 then give the answer as 7)

   Ans: 4

   $2n < n! \implies 2 < (n-1)! \implies (n-1) \neq 1, (n-1) \neq 2$ Hence, $(n-1) = 3$, so $n >= 4$ is necessary condition. We observe that it is true for $n = 4$, and the rest is proved by induction

5. Given p, we want to prove q. Which of the following will suffice:

   (a) $\neg q \implies \neg p$

   (b) $p \wedge q \implies q$

   (c) $\neg p \wedge \neg q \implies p$

   (d) $\neg q \implies q$

   (e) $p \wedge \neg q \wedge r \implies \neg r$

   (f) none of these

   Ans: a and d. This can be solved using truth tables. (a) is correct as it is proof by contrapositive, and (d) is correct as it shows that $\neg q$ is not possible.

6. Given p, we want to prove q. Which of the following will suffice:

   (a) $\neg p \wedge q \implies p$

   (b) $p \vee \neg q \implies \neg p$

   (c) $p \wedge \neg q \implies (1 = 0) \vee p$

(d) $q \wedge p \implies (1 = 1)$

(e) none of these

Ans: e. This can be solved using truth tables just like the previous question.

7. The sum $\sum_{k=1}^{n}(1+2+\cdots+k)$ is a polynomial of what degree

   (a) 1

   (b) 2

   (c) 3

   (d) 4

   (e) 5

   Ans: c

   We shall prove that $\sum_{k=1}^{n}(1+2+\cdots+k)$ is bounded above and below by a polynomial of degree three.

   $\sum_{k=1}^{n}(1+2+\cdots+k) \leq \sum_{k=1}^{n}(k+k+\cdots+k) = \sum_{k=1}^{n} k^2 \leq \sum_{k=1}^{n} n^2 = n^3$

   $\sum_{k=1}^{n}(1+2+\cdots+k) \geq \sum_{k=1}^{n}((k/2+1)+(k/2+2)+\cdots+(k/2+k/2)) \geq \sum_{k=1}^{n}(k/2+k/2+\cdots+k/2) \geq \sum_{k=1}^{n} k^2/4 \geq ((n/2)^2+(n/2)^2+\cdots+(n/2)^2)/4 = (n/2)^3/4 = n^3/32$

   Hence, the sum is a polynomial of degree three.

8. If $k > 1$ then $2^k - 1$ is not a perfect square. Which of the following is a correct proof?

   (a) If $2^k - 1 = n^2$ then $2^{k-1} - 1 = (n-1)^2$ and $(n^2+1)/[(n-1)^2+1] = 2^k/2^{k-1} = 2$. But this later ratio is 2 if and only if n=1 or n=3. Thus, $2^k - 1 = n^2$ leads to a contradiction.

   (b) If $2^k - 1 = n^2$ then $2^k = n^2 + 1$. Since 2 divides n^2 and 2 does not divide $n^2 + 1$. This is a contradiction since obviously 2 divides 2^k.

   (c) $2^k - 1$ is odd and an odd number can never be a perfect square.

   (d) If $2^k - 1 = n^2$ then n is odd. Since n is odd, therefore $2^k - 1 = 4j + 1$, which implies 2^k for all $k > 1$ is divisible by 2 but not by 4. This is a contradiction.

   (e) None of these.

   Ans: d. The first three proofs are wrong and (d) is a valid proof.

9. Sai wants to prove $\sqrt{2}$ is irrational. He decides to prove this by contradiction. So he assumes: $\sqrt{2}$ is rational and thus of the form p/q (Where p and q are integers). Which of the following proofs is/are correct:

   (a) Sai further assumes p is not divisible by 2 and proves that 2 divides p. Thus leading to a contradiction.

   (b) Sai further assumes p and q are coprime and proves that p and q are not coprime. Thus leading to a contradiction.

(c) $\sqrt{2}$ is irrational. In the above 2 options, Sai has made more than one assumption while using the proof by contradiction method. This is wrong and therefore the above 2 proofs are wrong.

(d) We cannot use proof by contradiction to prove this result.

Ans: b. The question already assumes that $\sqrt{2}$ is irrational, and hence there cannot be a second assumption that p is not divisible by two, hence (a) is not valid. The assumption that p and q are coprime is without loss of generality since such p and q can always be found.

10. If n is a positive natural number then

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)\times(2n+1)}$$

is equal to

(a) $2n/(n+1)$

(b) $n/(2n+1)$

(c) $n/(n+2)$

(d) $n/2(n+1)$

Ans: b

Proof by induction or by telescopic method

11. Let $P(n)$ be a statement and we prove $P(k) \implies P(k^2)$ and $P(k) \implies P(2k)$. Then we to prove that $P(n)$ is true for all n

(a) it is enough to prove the base case for $k = 1$

(b) it is enough to prove the base case for $k = 1$ and $k = 2$.

(c) it is enough to prove that base case for $k = 1$ and $k = 2$ and $k = 3$

(d) No base case can prove the statement.

Ans: d. As one cannot prove $k = 5$ by (a),(b) or (c)

12. The sum $\sum_{k=1}^{n}(1^3 + 2^3 + \cdots + k^3)$ is a polynomial of what degree

(a) 3

(b) 4

(c) 5

(d) 6

Ans:(c). Similar as problem 7.

13. Let $P(n)$ be a statement and we prove $P(k) \implies P(k^2)$ and $P(k) \implies P(k+3)$. Then we to prove that $P(n)$ is true for all n

(a) it is enough to prove the base case for $k = 1$

(b) it is enough to prove the base case for $k = 1$ and $k = 2$.

(c) it is enough to prove that base case for $k = 1$ and $k = 2$ and $k = 3$

(d) No base case can prove the statement.

Ans: c. (a) and (b) are false as this does not prove for $k = 3$. (c) is correct as the three base cases are sufficient for $P(k) \implies P(k+3)$ to imply the statement.

14. We are going to prove by induction that $\sum_{i=1}^{n} Q(i) = n^2(n+1)$. For which choice of $Q(i)$ will induction work?

(a) $3i^2 - 2$

(b) $2i^2$

(c) $3i^3 - i$

(d) $i(3i - 1)$

(e) $3i^3 - 7i$

Ans: d. By substitution, induction or summation formula. (Summation can be proved by telescopic sum method)

15. Let $P(n)$ be a statement and we prove $P(k) \implies P(k-3)$ and $P(k) \implies P(2k)$. Then we to prove that $P(n)$ is true for all n

(a) it is enough to prove the base case for $k = 1$

(b) it is enough to prove the base case for $k = 1$ and $k = 2$.

(c) it is enough to prove that base case for $k = 1$ and $k = 2$ and $k = 3$

(d) No base case can prove the statement.

Ans: (d),(a) and (b) are false as they do not imply $k = 3$. For d,

The statement is true for all numbers of the form 2^j and 3×2^j as $P(k) \implies P(2k)$. For all other numbers, we have:

Case I: $n \equiv 1 \pmod 3$

Consider some even power of 2, greater than n. This is of the form $3j + 1$, hence it implies n as $P(k) \implies P(k-3)$

Case II: $n \equiv 2 \pmod 3$

Consider some odd power of 2, greater than n. This is of the form $3j + 2$, hence it implies n as $P(k) \implies P(k-3)$

Case III: $n \equiv 0 \pmod 3$

Consider some number of the form 3×2^j greater than n. This implies n as $P(k) \implies P(k-3)$